

Canonical moments and random spectral measures

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Abstract

We study some connections between the random moment problem and the random matrix theory. A uniform draw in a space of moments can be lifted into the spectral probability measure of the pair (A, e) where A is a random matrix from a classical ensemble and e is a fixed unit vector. This random measure is a weighted sampling among the eigenvalues of A . We also study the large deviations properties of this random measure when the dimension of the matrix increases. The rate function for these large deviations involves the reversed Kullback information.

Key words: Random matrices, unitary ensemble, Jacobi ensemble, spectral measure, canonical moments, large deviations.

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1 Introduction

The aim of this paper is to establish a strong link between random moment problems ([5],[15]) and random matrices.

As a matter of fact, in the last decade, emphasis has been put on the asymptotic behavior of large random matrices. Such a study was first motivated by problems arising in theoretical physics ([29], [14]). Generally, the distribution of these random matrices is characterized by some invariance properties. In the GOE_N (resp. GUE_N), the matrix A_N is $N \times N$ symmetric

(resp. Hermitian) and, except for this constraint, has independent Gaussian entries. In the CUE_N , A_N has the Haar distribution on $\mathbb{U}(N)$ (the group of $N \times N$ unitary matrices). We refer to [1] and [21] for a general overview on the subject. In these examples, the distribution of both the random eigenvalues $\lambda_1, \dots, \lambda_N$ and eigenvectors are precisely known. These eigenvalues are (stochastically) independent of the eigenvectors and the matrix of the normalized eigenvectors is Haar distributed. The main object of interest is the empirical spectral distribution (ESD) of A_N , which is the discrete measure

$$\nu^{(N)} := \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}. \quad (1)$$

When A_N is drawn randomly, the asymptotic properties of the sequence of random probability measures $(\nu^{(N)})_N$ are obtained by a moment method using

$$\int x^k \nu^{(N)}(dx) = \frac{1}{N} \text{tr}(A_N^k), \quad (k \geq 1), \quad (2)$$

and $(\nu^{(N)})_N$ converges towards a deterministic probability measure. Namely, for the GOE (resp. GUE) the limit is the semicircle distribution while for the CUE, it is the uniform measure on the unit circle \mathbb{T} . To give the rate of convergence of the sequence $(\nu^{(N)})_N$, it is interesting to know if it satisfies the large deviation principle (LDP). This problem has been first studied for the GOE in the pioneer work of Ben Arous and Guionnet ([3]). It was further investigated for the CUE (see [20]) and many other distributions (see [13]). In these LDPs the speed is N^2 and the rate function may be obtained from the Voiculescu entropy ([21]).

In this paper, we consider another approach to unitary matrices. Instead of considering the classical empirical measure $\nu^{(N)}$ defined in (1), we study the random spectral measures (see (4) below). Every unitary operator A on a Hilbert space with a unit cyclic vector e is (by the spectral theorem) unitarily equivalent to the multiplication by z on $L^2(\mathbb{T}, \mu)$ for some probability measure μ , which is uniquely determined by its moments. If we endow $\mathbb{U}(N)$ with the normalized Haar measure $\lambda_{\mathbb{U}(N)}$, the first vector of the standard basis e_1 is a.s. cyclic for A_N . We thus define the random spectral measure $\mu_{\mathbf{w}}^{(N)}$ on \mathbb{T} related to the pair (A_N, e_1) . It is defined by its moments:

$$\int_{\mathbb{T}} z^k \mu_{\mathbf{w}}^{(N)}(dz) = \langle e_1, A_N^k e_1 \rangle, \quad (k \geq 1). \quad (3)$$

The ESD $\nu^{(N)}$ and the spectral measure $\mu_{\mathbf{w}}^{(N)}$ are widely different. This can be seen in terms of moments. In both cases, we consider successive powers of A_N , but the moments of $\mu_{\mathbf{w}}^{(N)}$ are the first entries given by (3) and those of $\nu^{(N)}$ are the normalized traces given by (2). Alternatively, a simple diagonalization of A_N gives:

$$\mu_{\mathbf{w}}^{(N)} := \sum_{j=1}^N \mathbf{w}_j \delta_{e^{i\theta_j}}, \quad (4)$$

where the \mathbf{w}_j 's are the square moduli of the top entries of the normalized unit eigenvectors and $\exp(i\theta_j) = \lambda_j$ are the (a.s. different) eigenvalues of A_N . It is now clear that this measure carries more information on A_N than $\nu^{(N)}$. Indeed, it involves weights depending on the eigenvectors of this matrix. In this paper, we are concerned with a precise study of the random probability measure $\mu_{\mathbf{w}}^{(N)}$.

For a probability measure ξ on \mathbb{T} , two encodings are interesting. First, its Verblunsky coefficients appear in the Schur recursion formula satisfied by the sequence of orthogonal polynomials in $L^2(\xi)$ (see for example [34], [35]). Secondly, its canonical moments are involved in statistical applications. They are inductively defined : the $(k+1)$ -th canonical moment is the relative position of the $(k+1)$ -th moment consistent with the k first ones (see [11] Chapter 9 for an overview on canonical moments).

Two remarkable papers motivated our work. On the one hand, Killip and Nenciu [26] proved that in the CUE_N model, the Verblunsky coefficients of $\mu_{\mathbf{w}}^{(N)}$ are independent and have known distributions (namely complex beta distribution). On the other hand, Lozada [27] has studied the distribution of the N first canonical moments for special random measures, namely the measures having their N first moments uniformly distributed. Surprisingly, the distributions found by Killip and Nenciu and by Lozada are the same. We can explain this last result by the fact that Verblunsky coefficients and canonical moments coincide (as pointed out by several authors, see Simon [33] p. 439). Using this observation we show in Theorem 4.1 that the vector of moments of $\mu_{\mathbf{w}}^{(N)}$ under CUE_N is also uniformly distributed, giving a (new) enlightening connection between random moment theory and random matrix theory.

Extending this method, we give the precise distributions of moments and canonical moments in other circular ensembles and in the so-called β -ensembles (or log-gases). Furthermore, we consider real matrices. Beginning

with the special orthogonal group of order $2N$ $\mathrm{SO}(2N)$, the symmetry property of spectral measures allows to consider their projections on $[-1, 1]$, which leads naturally to the family of Jacobi ensembles. We prove a lift of a uniform random point in the space of moments into the spectral measure of a random element of the ensemble $\mathrm{SO}(2N)/\mathrm{U}(N)$ (Theorem 4.10).

The common feature of all these models is that the weights and the support points of $\mu_{\mathbf{w}}^{(N)}$ are independent. It is a nice framework to study the LDP, with two main differences with the above mentioned LDP for ESD : the (random) weighting both slows down the speed from N^2 to N and (consequently) changes the rate function to the reversed Kullback information. This representation allows in particular to recover results proved by using the independence of canonical moments in [15] and [27]. As a consequence, we see that $\mu_{\mathbf{w}}^{(N)}$ converges weakly in probability to the same deterministic limits as the ESD $\nu^{(N)}$.

Some results for ensembles with non compactly supported spectral measures can be found in [2] for sample covariance matrices, and in [16] for the GUE.

The paper is organized as follows. In Section 2, we give some notations and list some distributions of frequent use. In Section 3, we explain the set-up of moments and canonical moments in the complex and real case. Section 4 is devoted to the distribution of random spectral measures in different models. Finally, in Section 5, we state LDPs for the families of random spectral measures.

Notice that a few months after dropping the first version of the present paper on Arxiv, we have been aware of the work of Birke and Dette [4] on the asymptotic behavior of the roots of random orthogonal polynomials associated to random moments. Their main result is the computation of the root distribution when the moments are uniformly distributed. It appears to be a particular case of our more general Lemma 4.7 and Theorem 4.8.

2 Notations and some useful distributions

Let \mathbb{D} be the open unit disk in \mathbb{C} and $\mathbb{T} = \partial\mathbb{D}$, the unit circle parameterized by $z = e^{i\theta}$, with $\theta \in [-\pi, \pi)$. Let μ be a probability measure on \mathbb{T} . μ is said to be trivial if it supported on a finite set, and nontrivial otherwise. Further, the Hermitian product on $L^2(\mathbb{T}, \mu)$ is defined by $\langle f, g \rangle := \int_{\mathbb{T}} \bar{f} g \, d\mu$. Let (G, \mathcal{G}) be any measurable space, we denote by $\mathcal{M}_1(G)$ (resp. $\mathcal{M}(G)$) the

set of all probability measures (resp. positive measures) on G . If $\mu \in \mathcal{M}_1(G)$ and f is integrable, we write sometimes $\mu(f)$ for $\int_G f \, d\mu$. We recall now some special useful distributions that are used later. For $k \geq 1$, we set

$$\begin{aligned}\mathcal{S}_k &:= \{(x_1, \dots, x_k) : x_i > 0, (i = 1, \dots, k), x_1 + \dots + x_k = 1\} \\ \mathcal{S}_k^< &:= \{(x_1, \dots, x_k) : x_i > 0, (i = 1, \dots, k), x_1 + \dots + x_k < 1\}.\end{aligned}$$

Obviously, the mapping $(x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, x_k)$ is one to one from the simplex \mathcal{S}_{k+1} onto $\mathcal{S}_k^<$.

For $a_j > 0$, $j = 1, \dots, k+1$, the Dirichlet distribution $\text{Dir}(a_1, \dots, a_{k+1})$ on \mathcal{S}_{k+1} has the density

$$\frac{\Gamma(a_1 + \dots + a_{k+1})}{\Gamma(a_1) \dots \Gamma(a_{k+1})} x_1^{a_1-1} \dots x_{k+1}^{a_{k+1}-1} \quad (5)$$

with respect to the Lebesgue measure on \mathcal{S}_{k+1} . When $a_1 = \dots = a_{k+1} = a > 0$, we denote the Dirichlet distribution by $\text{Dir}_{k+1}(a)$. If $a = 1$ we recover the uniform distribution on \mathcal{S}_{k+1} .

Pushing the Dirichlet distribution under the previous mapping, we get a probability measure $\text{Dir}(a_1, \dots, a_k; a_{k+1})$ on $\mathcal{S}_k^<$ with density :

$$\frac{\Gamma(a_1 + \dots + a_{k+1})}{\Gamma(a_1) \dots \Gamma(a_{k+1})} x_1^{a_1-1} \dots x_k^{a_k-1} (1 - x_1 - \dots - x_k)^{a_{k+1}-1}$$

If $k = 1$, $\text{Dir}(a; b)$ has the Beta(a, b) density on $[0, 1]$:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}.$$

The very particular case Beta($1/2, 1/2$) is the so-called arcsine law. Sometimes we need the distribution obtained by pushing forward Beta(a, b) under the mapping $x \mapsto 2x - 1$. It is the distribution Beta_s(b, a) on $(-1, 1)$ having density

$$2^{1-a-b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (1+x)^{a-1} (1-x)^{b-1}.$$

We use also a complex version of the beta distribution. For $r > -1$ let η_r be the probability density on \mathbb{C} defined by

$$\eta_r(z) := \frac{r+1}{\pi} (1 - |z|^2)^r, \quad (z \in \mathbb{D}). \quad (6)$$

It is obviously the density of $X = e^{iU} \sqrt{B}$ where U is uniform on $[0, 2\pi]$ and B is Beta(1, r) distributed.

To end this section, let us recall the classical relation between the Dirichlet and Gamma distributions. A Gamma variable γ_a with parameter $a > 0$ has density

$$\frac{t^{a-1} e^{-t}}{\Gamma(a)}, \quad (t > 0).$$

It is well known that if $y_i, i = 1, \dots, r$ are independent and if $y_i \stackrel{(d)}{=} \gamma_{b_i}$ ($b_i > 0$) then

$$\left(\frac{y_1}{y_1 + \dots + y_r}, \dots, \frac{y_r}{y_1 + \dots + y_r} \right) \stackrel{(d)}{=} \text{Dir}(b_1, \dots, b_r) \quad (7)$$

and this variable is independent of $y_1 + \dots + y_r$.

3 Verblunsky coefficients, canonical moments and uniform probability

We now present the notion of canonical moments of a measure. We successively focus on the complex case (unitary circle) and on the real case (compact interval). Finally we recall that the uniform probability on moment spaces corresponds to a nice distribution on the space of canonical moments.

3.1 The complex case : Verblunsky coefficients

All the material of this subsection comes from [34] Section 1 or [33] Sections 2 and 3. We recall here the connection between moments of a probability measure on \mathbb{T} and Verblunsky coefficients built through orthogonal polynomials.

Let μ be an arbitrary nontrivial probability measure on \mathbb{T} . The functions $1, z, z^2, \dots$ are linearly independent in $L^2(\mathbb{T}, \mu)$. Following the Gram-Schmidt procedure we define the infinite sequence $(\Phi_j)_{j \geq 1}$ of monic orthogonal polynomials. More precisely, $\Phi_0(z) \equiv 1$ and $\Phi_j(z)$ is the projection of z^j onto $\{1, \dots, z^{j-1}\}^\perp$, for $j \geq 1$.

If $\mu \in \mathcal{M}_1(\mathbb{T})$ has finite support $\{z_1, \dots, z_K\}$ (K different points), we still define Φ_j in the same way for $j = 1, \dots, K-1$. Besides, we define Φ_K

as the unique monic polynomial of degree K such that $\|\Phi_K\| = 0$ i.e.

$$\Phi_K(z) = \prod_{j=1}^K (z - z_j). \quad (8)$$

By convention, in the nontrivial case we set $K = \infty$, and then the sentence "for every $j < K$ " (or "for every $j < K + 1$ ") mean "for every finite j ".

Some useful polynomials associated to the sequence $(\Phi_j)_{j < K}$ are the reversed (or reciprocal) polynomials. They are defined by $\Phi_0^*(z) \equiv 1$ and

$$\Phi_j^*(z) = z^j \overline{\Phi_j(1/\bar{z})}. \quad (9)$$

In the next proposition, we define the Verblunsky coefficients. They are sometimes called Schur, Szegő or Geronimus coefficients or even reflection coefficients. Surprisingly, these coefficients also appears as central quantities in the theory of moment problems (see for example [11]), where they are called canonical moments.

Proposition 3.1 (Szegő) *For $j < K + 1$, we define the Verblunsky coefficient of order j by setting $c_j := -\overline{\Phi_j(0)}$. The sequence $(\Phi_j)_{j < K+1}$ satisfies the recursion*

$$\Phi_0 = 1 \quad , \quad \Phi_j = z\Phi_{j-1} - \bar{c}_j\Phi_{j-1}^* \quad (1 \leq j < K + 1).$$

Notice that if μ is nontrivial, $c_j \in \mathbb{D}$ for every $j > 0$. If $K < \infty$, then $c_j \in \mathbb{D}$ for $1 \leq j \leq K - 1$ and $c_K \in \mathbb{T}$ (see [34] Theorem 1.5.2). In the sequel, when we need to stress the dependence of the Verblunsky coefficients on the underlying measure μ , we write $c_j(\mu)$. A theorem due to Verblunsky claims that the correspondence between a probability measure on \mathbb{T} and the sequence of its coefficients is one to one ([34] Theorem 1.7.11). For $N \geq 1$, set

$$M_N^{\mathbb{T}} = \left\{ \left(\int_{\mathbb{T}} z^j \mu(dz) \right)_{1 \leq j \leq N} : \mu \in \mathcal{M}_1(\mathbb{T}) \right\}. \quad (10)$$

Proposition 3.2 ([34] pp. 60 and 218, [11] p.269) *Let $(t_1, \dots, t_N) \in \text{int } M_N^{\mathbb{T}}$, the range of the $(N + 1)$ th moment*

$$t_{N+1} = \int_{\mathbb{T}} z^{N+1} \eta(dz)$$

as η varies over all probability measures having (t_1, \dots, t_N) as their N first moments, is a disk centered at some point s_N (depending on t_1, \dots, t_N) with radius

$$r_N = \prod_{j=1}^N (1 - |c_j|^2) \neq 0$$

The relative position is

$$\frac{t_{N+1} - s_N}{r_N} \in \mathbb{D},$$

and it is exactly \bar{c}_{N+1} .

This striking connection of two notions (related position versus recursion coefficient) has been proved in two opposite ways: starting from the recursion to get relative position [34] p. 60), or alternatively starting from the moments and their relative positions and computing the recursion [34] p. 218, [11] p.269. For an historical account, see [34] p.10 and p.221.

3.2 Canonical moments : Real case

Following the last interpretation of canonical moments as relative positions of moments in $M_N^{\mathbb{T}}$, it is possible to build similar quantities when the set of integration \mathbb{T} is replaced by a compact interval $[a, b]$ ($a < b$). Let us briefly recall the exact definition of canonical moments in this frame. We refer to the excellent book of Dette and Studden [11] for a complete overview on the subject. To begin, for $N \geq 1$, we denote by $M_N^{[a,b]}$ the N -th moment space generated by probability measures on $[a, b]$:

$$M_N^{[a,b]} := \left\{ \left(\int_a^b x^k \mu(dx), k = 1, \dots, N \right) : \mu \in \mathcal{M}_1([a, b]) \right\}.$$

Given $\mathbf{m}^j := (m_1, \dots, m_j) \in \text{int } M_j^{[a,b]}$, ($j \geq 1$), we first define the *extreme* values,

$$m_{j+1}^+(\mathbf{m}^j) = \max \left\{ m \in \mathbb{R} : (m_1, \dots, m_j, m) \in M_{j+1}^{[a,b]} \right\} \quad (11)$$

$$m_{j+1}^-(\mathbf{m}^j) = \min \left\{ m \in \mathbb{R} : (m_1, \dots, m_j, m) \in M_{j+1}^{[a,b]} \right\}. \quad (12)$$

For $k \geq i \geq 1$, the i -th canonical moment is defined recursively as

$$c_i(\mathbf{m}^k) = c_i(\mathbf{m}^i) := \frac{m_i - m_i^-(\mathbf{m}^{i-1})}{m_i^+(\mathbf{m}^{i-1}) - m_i^-(\mathbf{m}^{i-1})} \quad (13)$$

A quite nice property of canonical moments is that they are invariant on any affine one to one mapping transforming the support of the underlying measures (see for example [11]). So that, we may restrict ourselves to the special case of $M_N^{[0,1]}$.

3.3 Uniform probability on moment spaces

In this subsection, we endow the moment sets $M_n^{\mathbb{T}}$ and $M_n^{[0,1]}$ (for fixed n) with the uniform probability and recall that in these cases the canonical moments previously defined have very interesting properties (see Lemmas 3.4 and 3.3 below).

As pointed in Section 3.1, for fixed N there is a one to one mapping between moments and canonical moments. Indeed, we may define a one to one mapping $\kappa_N^{\mathbb{T}}$

$$\begin{aligned} \kappa_N^{\mathbb{T}} : \text{int } M_N^{\mathbb{T}} &\rightarrow \mathbb{D}^N \\ (t_1, \dots, t_N) &\mapsto (c_1, \dots, c_N). \end{aligned} \quad (14)$$

In the real case, the relation (13) defines a one to one mapping $\kappa_N^{[0,1]}$

$$\begin{aligned} \kappa_N^{[0,1]} : \text{int } M_N^{[0,1]} &\rightarrow (0, 1)^N \\ (t_1, \dots, t_N) &\mapsto (c_1, \dots, c_N). \end{aligned} \quad (15)$$

which is *triangular* and bijective.

The two following lemmas give the canonical moment distribution when the moments are uniformly drawn.

Lemma 3.3 (Lozada) *Endowing $\text{int } M_N^{\mathbb{T}}$ with the uniform distribution is equivalent to the N first canonical moments (c_1, \dots, c_N) being independent in such a way that c_j has density η_{N-j} .*

Lemma 3.4 (Chang, Kemperman, Studden) *Endowing $\text{int } M_N^{[0,1]}$ with the uniform distribution is equivalent to the N canonical moments (c_1, \dots, c_N) being independent in such a way that c_j is $\text{Beta}(N - j + 1, N - j + 1)$ distributed.*

These two lemmas have been the starting point for the investigation on the asymptotic behavior of the randomized sets of moments $M_N^{[0,1]}$ and $M_N^{\mathbb{T}}$ (see

[5], [15], [27] and [10]). Here, these lemmas are useful to obtain some very nice elementary properties of random measures built on eigenvalues of some classical random matrix models. We develop these results in the next sections.

4 Random spectral measures and their moments

In this Section, we first define the spectral measure associated with the pair (A, e) where A is a unitary matrix and e a cyclic vector. Then we randomly draw a matrix and study the distribution of the associated random spectral measure and of its canonical moments. We focus on various classical distributions popular in the random matrix paradigm. Moreover we extend the class of such random measures leading to log-gases models, which in turn may be lifted into matrix ensembles. In particular, we emphasize the different ways to get uniform distribution on the space of moments. Our main results in this section are Theorem 4.1, Theorem 4.8 and Theorem 4.10.

4.1 Spectral measures associated with a unitary matrix

Let us consider \mathbb{C}^N with its canonical basis (e_1, \dots, e_N) and $\mathbb{U}(N)$ the group of $N \times N$ unitary matrices. For $A \in \mathbb{U}(N)$, let λ_i , $i = 1, \dots, N$ be the eigenvalues of A . We may write

$$A = \Pi D \Pi^*, \quad (16)$$

where $D := \text{diag}(\lambda_i)_{i=1, \dots, N}$ is diagonal and $\Pi := (\pi_{ij})_{i,j=1, \dots, N}$ is unitary. Let us assume that e_1 is cyclic, i.e. that $\text{span}(e_1, Ae_1, \dots, A^{N-1}e_1)$ has rank N . Following [35], we consider the spectral measure $\mu_{\mathbf{w}}^{(N)}$ associated with the pair (A, e_1) i.e. having, for any $k \in \mathbb{N}$, moment of order k equal to $\langle e_1, A^k e_1 \rangle$. Obviously, $\mu_{\mathbf{w}}^{(N)}$ is trivial and supported by the eigenvalues of A . More precisely,

$$\mu_{\mathbf{w}}^{(N)} = \sum_k \mathbf{w}_k \delta_{\lambda_k}, \quad (17)$$

where $\mathbf{w}_k := |\pi_{1k}|^2$, $k = 1, \dots, N$ are the square moduli of the top entries of the normalized eigenvectors.

4.2 Circular ensembles

Here are recalled three families of classical ensembles of random matrices:

- CUE_N (circular unitary ensemble) : $\mathbb{U}(N)$ equipped with its normalized Haar measure $\lambda_{\mathbb{U}(N)}$
- COE_N (circular orthogonal ensemble) : set of symmetric unitary $N \times N$ matrices. Every element S can be written $S = g^T g$ with $g \in \mathbb{U}(N)$ hence $\text{COE}_N \cong \mathbb{U}(N)/\mathbb{O}(N)$. The probability measure is the pushforward of $\lambda_{\mathbb{U}(N)}$ under the projection mapping $\mathbb{U}(N) \rightarrow \mathbb{U}(N)/\mathbb{O}(N)$.
- CSE_N (circular symplectic ensemble) : set of $2N \times 2N$ self-dual unitary matrices. Recall that the dual of a matrix H is $H^D := JH^T J^T$ with

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

and that a $2N \times 2N$ matrix k is symplectic if it satisfies $kJk^T = J$. Every element \tilde{S} can be written $\tilde{S} = g^D g$ with $g \in \mathbb{U}(2N)$, hence $\text{CSE}_N \cong \mathbb{U}(2N)/\text{USp}(2N)$, where $\text{USp}(2N)$, the set of unitary symplectic matrices, is the invariant set of the involution $g \mapsto (g^D)^{-1}$. The probability measure is the pushforward of $\lambda_{\mathbb{U}(2N)}$ under the projection mapping $\mathbb{U}(2N) \rightarrow \mathbb{U}(2N)/\text{USp}(2N)$.

All these ensembles are considered in the two following subsections.

4.2.1 The circular unitary ensemble

Let us first notice that with probability 1, the vector e_1 is cyclic. If $A \in \mathbb{U}(N)$, all its eigenvalues have unit modulus and in the decomposition (16) we may write $D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$. Further, the matrix Π in (16) is not uniquely determined. But, it is well known that it can be chosen such that, if A is Haar distributed then Π is also Haar distributed and independent of D . In this case, the joint law of eigenvalues arguments (on $[0, 2\pi]^N$) is

$$\text{CUE}_N(d\theta_1, \dots, d\theta_N) := \frac{1}{N!(2\pi)^N} |\Delta(e^{i\theta_1}, \dots, e^{i\theta_N})|^2 d\theta_1 \cdots d\theta_N \quad (18)$$

where Δ is the Vandermonde determinant

$$\Delta(x_1, \dots, x_N) = \prod_{1 \leq j < k \leq N} (x_j - x_k) . \quad (19)$$

Furthermore, since Π is Haar distributed, its first row is uniformly distributed on the Euclidean sphere of \mathbb{C}^N . So that, setting $\mathbf{w}_k := |\pi_{1,k}|^2$, ($k = 1, \dots, N$) the vector $(\mathbf{w}_1, \dots, \mathbf{w}_N)$ is uniformly distributed on \mathcal{S}_N (see for example Proposition 3.1 of [26]) i.e.

$$(\mathbf{w}_1, \dots, \mathbf{w}_N) \stackrel{(d)}{=} \text{Dir}_N(1) \quad (20)$$

(see (5) for the definition).

So, from (17) we have built the random probability measure on \mathbb{T} :

$$\mu_{\mathbf{w}}^{(N)} = \sum_{k=1}^N \mathbf{w}_k \delta_{e^{i\theta_k}}, \quad (21)$$

supported by the eigenvalues of the matrix A with an independent system of weights.

Up to our knowledge, the following result was not known previously.

Theorem 4.1 *Under CUE_N , the distribution of $\left(\int_{\mathbb{T}} z^j \mu_{\mathbf{w}}^{(N)}(dz)\right)_{1 \leq j \leq N-1}$ is uniform on $M_{N-1}^{\mathbb{T}}$.*

Proof: Let $(c_1(\mu_{\mathbf{w}}^{(N)}), \dots, c_N(\mu_{\mathbf{w}}^{(N)}))$ be the vector of N first canonical moments of $\mu_{\mathbf{w}}^{(N)}$. In Proposition 3.3 of [26] (see also [36] Section 11) it is proved that the components of this vector are independent. Furthermore,

$$\text{CUE}_N(c_j(\mu_{\mathbf{w}}^{(N)}) \in dz) = \eta_{N-j-1}(z) dz, \quad j = 1, \dots, N-1, \quad (22)$$

and c_N is uniform on \mathbb{T} (since the support of $\mu_{\mathbf{w}}^{(N)}$ is a finite set, the last canonical moment belongs to \mathbb{T}). Considering now the pushforward of these distributions under $(\kappa_{N-1}^{\mathbb{T}})^{-1}$ the measurable one to one mapping from \mathbb{D}^{N-1} to $\text{int } M_{N-1}^{\mathbb{T}}$ which map canonical moments to moments we may conclude using Lemma 3.3. \blacksquare

For the special unitary group $\text{SU}(N)$ we also have.

Corollary 4.2 *Under the normalized Haar measure $\lambda_{\text{SU}(N)}$ on $\text{SU}(N)$, the distribution of $\left(\int_{\mathbb{T}} z^j \mu_{\mathbf{w}}^{(N)}(dz)\right)_{1 \leq j \leq N-1}$ is uniform on $M_{N-1}^{\mathbb{T}}$.*

Proof: The joint eigenvalue distribution on \mathbb{T}^{N-1} of the normalized Haar measure on $\mathrm{SU}(N)$ is

$$\frac{1}{N!(2\pi)^{N-1}} |\Delta(e^{i\theta_1}, \dots, e^{i\theta_N})|^2 d\theta_1 \cdots d\theta_{N-1}$$

with $\theta_N = -(\theta_1 + \cdots + \theta_{N-1}) \pmod{2\pi}$ (see [23]). Notice that it is the density of $(\theta_1, \dots, \theta_{N-1})$ under CUE_N conditioned on $\theta_N = -(\theta_1 + \cdots + \theta_{N-1}) \pmod{2\pi}$. Moreover we have $\exp(i(\theta_1 + \cdots + \theta_N)) = (-1)^N \Phi_N(0) = (-1)^{N+1} \bar{c}_N$, so that the conditioning set is $\{c_N = (-1)^{N+1}\}$. We have seen in the proof of Theorem 4.1 that c_1, \dots, c_N are independent under CUE_N . Therefore, c_1, \dots, c_{N-1} remain independent when conditioning on $\{c_N = (-1)^{N+1}\}$. Moreover, their distributions are not affected by this conditioning. Using Lemma 3.3 we deduce that the conclusion of the Corollary holds true. ■

4.2.2 The circular β -models

For the COE and CSE we have also independence between the weights and the support points. Moreover, the distributions are given by formulas analogous to (18) and (20). Actually it is convenient (and now classical) to define a class of distributions with a continuous parameter $\beta > 0$, extending the cases $\beta = 1$ (COE), $\beta = 2$ (CUE) and $\beta = 4$ (CSE).

Definition 4.3 For $\beta > 0$, let $C\beta E_N$ the distribution on $\mathbb{T}^N \times \mathcal{S}_N$ such that

1) The joint law of $(\theta_1, \dots, \theta_N)$ is :

$$C\beta E_N(d\theta_1, \dots, d\theta_N) := C_\beta(N)^{-1} |\Delta(e^{i\theta_1}, \dots, e^{i\theta_N})|^\beta d\theta_1 \cdots d\theta_N,$$

where

$$C_\beta(N) = (2\pi)^N \frac{\Gamma(1 + \frac{\beta N}{2})}{(\Gamma(1 + \frac{\beta}{2}))^N}.$$

2) The joint distribution of $(\mathbf{w}_1, \dots, \mathbf{w}_N)$ on \mathcal{S}_N is $\mathrm{Dir}_N(\beta/2)$.

3) The variables $\theta_1, \dots, \theta_N$ and $\mathbf{w}_1, \dots, \mathbf{w}_N$ are independent.

Setting

$$\mu_{\mathbf{w}}^{(N)} = \sum_{k=1}^N \mathbf{w}_k \delta_{e^{i\theta_k}} \quad (23)$$

[26] Killip and Nenciu proved in their Proposition 4.2 that under $C\beta E_N$:

- 1) the Verblunsky coefficients $c_j = c_j(\mu_{\mathbf{w}}^{(N)})$, $(j = 1, \dots, N)$ are independent,
- 2) for $j = 1, \dots, N - 1$,

$$C\beta E_N(c_j(\mu_{\mathbf{w}}^{(N)}) \in dz) = \eta_{\frac{\beta}{2}(N-j)-1}(z) dz, \quad (24)$$

- 3) the distribution of $c_N(\mu_{\mathbf{w}}^{(N)})$ is uniform on \mathbb{T} .

Actually, they built also an explicit random matrix model (namely a five-diagonal matrix) whose spectral elements $(e^{i\theta_1}, \dots, e^{i\theta_N})$ and $(\mathbf{w}_1, \dots, \mathbf{w}_N)$ satisfy the previous properties.

4.3 $\mathbb{SO}(2N)$ and Jacobi ensembles

In this subsection, we consider real random matrices. We first consider the subgroup $\mathbb{SO}(2N)$ of $\mathbb{U}(2N)$, for which spectral measures (on \mathbb{T}) are symmetric (i.e. invariant by complex conjugation). This entails that the Verblunsky coefficients are real. We recall the results of [26] on the distributions of these coefficients, for $\mathbb{SO}(2N)$ equipped with the Haar measure and also for a three-parameter family of log-gases. Then, we project such spectral measures on \mathbb{R} and, by tuning the parameters we obtain random probability measures on $[0, 1]$, whose moments are uniform.

4.3.1 $\mathbb{SO}(2N)$ and extension

If we provide $\mathbb{SO}(2N)$ with the normalized Haar measure $\lambda_{\mathbb{SO}(2N)}$, then ± 1 are a.s. not eigenvalues and the spectral measure defined in (17) is a.s. supported by pairwise conjugate complex numbers. It may be written as:

$$\mu_{\mathbf{w}}^{(2N)} = \sum_{k=1}^N \frac{\mathbf{w}'_k}{2} (\delta_{e^{i\theta_k}} + \delta_{e^{-i\theta_k}}). \quad (25)$$

In that case, the distribution of $(\theta_1, \dots, \theta_N)$ has a density proportional to

$$|\Delta(2 \cos \theta_1, \dots, 2 \cos \theta_N)|^2 \quad (26)$$

and the array of weights $(\mathbf{w}'_1, \dots, \mathbf{w}'_N)$ is independent of $(\theta_1, \dots, \theta_N)$ and satisfies

$$(\mathbf{w}'_1, \dots, \mathbf{w}'_N) \stackrel{(d)}{=} \text{Dir}_N(1),$$

(see [26] Proposition 3.4).

A natural generalization is obtained replacing the square in (26) by an exponent β as in Section 4.2.2 and adding a particular external potential:

Definition 4.4 *For $a, b, \beta > 0$ let $\tilde{J}(\beta, a, b, N)$ be the distribution of $(\theta_1, \dots, \theta_N)$ having a density proportional to*

$$|\Delta(\cos \theta_1, \dots, \cos \theta_N)|^\beta \prod_{k=1}^N (1 - \cos \theta_k)^{a-\frac{1}{2}} (1 + \cos \theta_k)^{b-\frac{1}{2}}. \quad (27)$$

In this generalization, the random canonical moments have some very remarkable properties (like for the $C\beta E_N$ model) as soon as the weights are suitably distributed:

Proposition 4.5 *(Killip-Nenciu [26] Prop. 5.3)*

1. *For $\mathbb{SO}(2N)$ equipped with the Haar measure $\lambda_{\mathbb{SO}(2N)}$, the Verblunsky coefficients $c_1(\mu_{\mathbf{w}}^{(2N)}), \dots, c_{2N-1}(\mu_{\mathbf{w}}^{(2N)})$ are independent and satisfy:*

$$c_k(\mu_{\mathbf{w}}^{(2N)}) \stackrel{(d)}{=} \text{Beta}_s \left(\frac{2N-k}{2}, \frac{2N-k}{2} \right) \quad (k \leq 2N-1).$$

Moreover $c_{2N}(\mu_{\mathbf{w}}^{(2N)}) = -1$.

2. *More generally, under the distribution $\tilde{J}(\beta, a, b, N) \otimes \text{Dir}_N(\beta/2)$ on $[0, 2\pi]^N \times \mathcal{S}_N$, the Verblunsky coefficients $c_1(\mu_{\mathbf{w}}^{(2N)}), \dots, c_{2N-1}(\mu_{\mathbf{w}}^{(2N)})$ are independent and satisfy:*

$$\begin{aligned} c_k(\mu_{\mathbf{w}}^{(2N)}) &\stackrel{(d)}{=} \text{Beta}_s \left(\frac{2N-k-1}{4}\beta + a, \frac{2N-k-1}{4}\beta + b \right) \quad k \text{ odd} \\ c_k(\mu_{\mathbf{w}}^{(2N)}) &\stackrel{(d)}{=} \text{Beta}_s \left(\frac{2N-k-2}{4}\beta + a + b, \frac{2N-k}{4}\beta \right) \quad k \text{ even}, \end{aligned}$$

for $k \leq 2N-1$. Moreover $c_{2N}(\mu_{\mathbf{w}}^{(2N)}) = -1$.

It is then natural to project the symmetric measure $\mu_{\mathbf{w}}^{(N)}$ on \mathbb{R} . It is the motivation of the next subsection.

4.3.2 Jacobi ensembles

Let μ be a symmetric probability measure on \mathbb{T} (i.e. invariant under complex conjugation) and let $\gamma = R(\mu)$ be the pushforward of μ by the mapping $e^{i\theta} \mapsto \frac{1+\cos\theta}{2}$, so that for g continuous on $[0, 1]$

$$\int_0^1 g(x) d\gamma(x) = \int_{\mathbb{T}} g\left(\frac{2+z+\bar{z}}{4}\right) \mu(dz). \quad (28)$$

Applying that to a probability measure with finite support

$$\mu_{\mathbf{w}}^{(N)} = \sum_{k=1}^N \frac{\mathbf{w}'_k}{2} (\delta_{e^{i\theta_k}} + \delta_{e^{-i\theta_k}})$$

we get a probability measure

$$\gamma_{\mathbf{w}}^{(N)} = R(\mu_{\mathbf{w}}^{(N)}) = \sum_{k=1}^N \mathbf{w}'_k \delta_{x_k} \quad \text{where } x_k = \frac{1 + \cos \theta_k}{2}, \quad k = 1, \dots, N. \quad (29)$$

The correspondence between the Verblunsky/canonical coefficients of μ and the canonical coefficients of $\gamma = R(\mu)$ (in the sense of Subsection 3.2) is

$$c_k(\gamma) = \frac{1}{2} (1 + c_k(\mu)) \quad (\text{or, equivalently } c_k(\mu) = 2c_k(\gamma) - 1). \quad (30)$$

This claim is proved by a geometrical argument in [32] (Theorem 13.3.1) and using Tchebychev polynomials in [11] (Section 9.3.8).

Definition 4.6 For $a, b, \beta > 0$, let $J(\beta, a, b, N)$ the distribution on $[0, 1]^N$ with density proportional to

$$|\Delta(x_1, \dots, x_N)|^\beta \prod_{k=1}^N x_k^{b-1} (1 - x_k)^{a-1}.$$

If we equip $[0, 2\pi]^N$ with $\tilde{J}(\beta, a, b, N)$ then the joint distribution of x_1, \dots, x_n is $J(\beta, a, b, N)$. We are now ready to claim :

Lemma 4.7 Under the distribution $J(\beta, \frac{\beta}{4}, \frac{\beta}{4}, N) \otimes \text{Dir}_N(\beta/2)$, the canonical moments $c_1(\gamma_{\mathbf{w}}^{(N)}), \dots, c_{2N-1}(\gamma_{\mathbf{w}}^{(N)})$ are independent and

$$c_k(\gamma_{\mathbf{w}}^{(N)}) \stackrel{(d)}{=} \text{Beta}\left(\frac{2N-k}{4}\beta, \frac{2N-k}{4}\beta\right) \quad (k \leq 2N-1). \quad (31)$$

Proof: From (30) and (29) we have $c_k(\gamma_{\mathbf{w}}^{(N)}) = \frac{1}{2}[1 + c_k(\mu_{\mathbf{w}}^{(N)})]$. Applying Proposition 4.5 2), we see that the new coefficients inherit independence and have the mentioned distributions. ■

Theorem 4.8

1. If $\gamma_{\mathbf{w}}^{(N)} = \sum_{k=1}^N \mathbf{w}_k \delta_{x_k}$ with

$$(x_1, \dots, x_N, \mathbf{w}_1, \dots, \mathbf{w}_N) \stackrel{(d)}{=} J(4, 1, 1, N) \otimes \text{Dir}_N(2), \quad (32)$$

then the distribution of $(m_1(\gamma_{\mathbf{w}}^{(N)}), \dots, m_{2N-1}(\gamma_{\mathbf{w}}^{(N)}))$ is uniform on $M_{2N-1}^{[0,1]}$.

2. If $\gamma_{\mathbf{w}}^{(N)} = \mathbf{w}_0 \delta_0 + \sum_{k=1}^N \mathbf{w}_k \delta_{x_k}$

$$(x_1, \dots, x_N, \mathbf{w}_0, \dots, \mathbf{w}_N) \stackrel{(d)}{=} J(4, 1, 3, N) \otimes \text{Dir}(1, 2, \dots, 2, 2), \quad (33)$$

then the distribution of $(m_1(\gamma_{\mathbf{w}}^{(N)}), \dots, m_{2N}(\gamma_{\mathbf{w}}^{(N)}))$ is uniform on $M_{2N}^{[0,1]}$.

3. If $\gamma_{\mathbf{w}}^{(N)} = \mathbf{w}_0 \delta_0 + \sum_{k=1}^{N-1} \mathbf{w}_k \delta_{x_k} + \mathbf{w}_N \delta_1$ with

$$(x_1, \dots, x_{N-1}, \mathbf{w}_0, \dots, \mathbf{w}_N) \stackrel{(d)}{=} J(4, 3, 3, N-1) \otimes \text{Dir}(1, 2, \dots, 2, 1), \quad (34)$$

then the distribution of $(m_1(\gamma_{\mathbf{w}}^{(N)}), \dots, m_{2N-1}(\gamma_{\mathbf{w}}^{(N)}))$ is uniform on $M_{2N-1}^{[0,1]}$.

4. If $\gamma_{\mathbf{w}}^{(N)} = \sum_{k=1}^N \mathbf{w}_k \delta_{x_k} + \mathbf{w}_{N+1} \delta_1$ with

$$(x_1, \dots, x_N, \mathbf{w}_1, \dots, \mathbf{w}_{N+1}) \stackrel{(d)}{=} J(4, 3, 1, N) \otimes \text{Dir}(2, 2, \dots, 2, 1), \quad (35)$$

then the distribution of $(m_1(\gamma_{\mathbf{w}}^{(N)}), \dots, m_{2N}(\gamma_{\mathbf{w}}^{(N)}))$ is uniform on $M_{2N}^{[0,1]}$.

To prove the theorem we need the following proposition on principal representations of finite moment sequences (see for example [24], [25] or [11] Definition 1.2.10). These representations are related to the extreme values introduced in (11) and (12).

Proposition 4.9

1. Let $(m_1, \dots, m_{2N-1}) \in M_{2N-1}^{[0,1]}$

- *Lower principal representation: there exists a unique system $0 < x_1 < \dots < x_N < 1$ and $\mathbf{w}_1, \dots, \mathbf{w}_N \in (0, 1)$ with $\sum_{k=1}^N \mathbf{w}_k = 1$ such that the probability measure $\nu_- := \sum_{k=1}^N \mathbf{w}_k \delta_{x_k}$ satisfies*

$$m_j = \int_0^1 x^j \nu_-(dx), \quad j = 1 \dots 2N-1.$$

- *Upper principal representation: there exists a unique system $0 < x_1 < \dots < x_{N-1} < 1$ and $\mathbf{w}_0, \dots, \mathbf{w}_N \in (0, 1)$ with $\sum_{k=0}^N \mathbf{w}_k = 1$ such that the probability measure $\nu_+ := \mathbf{w}_0 \delta_0 + \sum_{k=1}^{N-1} \mathbf{w}_k \delta_{x_k} + \mathbf{w}_N \delta_1$ satisfies*

$$m_j = \int_0^1 x^j \nu_+(dx), \quad j = 1 \dots 2N.$$

2. Let $(m_1, \dots, m_{2N}) \in \text{int}(M_{2N}^{[0,1]})$

- *Lower principal representation: there exists a unique system $0 < x_1 < \dots < x_N < 1$ and $\mathbf{w}_0, \dots, \mathbf{w}_N \in (0, 1)$ with $\sum_{k=0}^N \mathbf{w}_k = 1$ such that the probability measure $\nu_- := \mathbf{w}_0 \delta_0 + \sum_{k=1}^N \mathbf{w}_k \delta_{x_k}$ satisfies*

$$m_j = \int_0^1 x^j \nu_-(dx), \quad j = 1 \dots 2N.$$

- *Upper principal representation: there exists a unique system $0 < x_1 < \dots < x_N < 1$ and $\mathbf{w}_1, \dots, \mathbf{w}_{N+1} \in (0, 1)$ with $\sum_{k=1}^{N+1} \mathbf{w}_k = 1$ such that the probability measure $\nu_+ := \sum_{k=1}^N \mathbf{w}_k \delta_{x_k} + \mathbf{w}_{N+1} \delta_1$ satisfies*

$$m_j = \int_0^1 x^j \nu_+(dx), \quad j = 1 \dots 2N.$$

Proof: [of Theorem 4.8]

- To prove the first point, take $\beta = 4$ in Lemma 4.7 and $n = 2N - 1$ in Lemma 3.4.
- To prove the remaining parts of the theorem we use Proposition 4.9. We only treat the point 2) since the proof of the other ones may be

tackled identically. From Proposition 4.9 (even case, lower representation) the mapping between $(x_1, \dots, x_N, \mathbf{w}_1, \dots, \mathbf{w}_N)$ and (m_1, \dots, m_{2N}) is a diffeomorphism. Thus the probability density of the moments are completely known once we compute the Jacobian. Hence, we have to compute

$$J = \begin{vmatrix} x_1 & \mathbf{w}_1 & x_2 & \mathbf{w}_2 & \cdots & x_N & \mathbf{w}_N \\ x_1^2 & 2\mathbf{w}_1 x_1 & x_2^2 & 2\mathbf{w}_2 x_2 & \cdots & x_N^2 & 2\mathbf{w}_N x_N \\ x_1^3 & 3\mathbf{w}_1 x_1^2 & x_2^3 & 3\mathbf{w}_2 x_2^2 & \cdots & x_N^3 & 3\mathbf{w}_N x_N^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{2N} & 2N\mathbf{w}_1 x_1^{2N-1} & x_2^{2N} & 2N\mathbf{w}_2 x_2^{2N-1} & \cdots & x_N^{2N} & 2N\mathbf{w}_N x_N^{2N-1} \end{vmatrix}$$

An easy calculation yields

$$J = \left(\prod_{k=1}^N \mathbf{w}_k \right) \left(\prod_{k=1}^N x_k^2 \right) \Delta(x_1, \dots, x_N)^4.$$

Up to a normalizing constant, we recognize the density of $J(4, 1, 3, N) \otimes \text{Dir}(1, 2, \dots, 2, 2)$. ■

From Lemma 3.4, the distribution of the canonical moments in the four cases of the previous theorem is completely known. In their Section 2, Killip and Nenciu [26] give a model of tridiagonal real matrices admitting these spectral characteristics: their spectral random measures have the same distribution as $\gamma_{\mathbf{w}}$. Here, we present for the first and fourth cases of Theorem 4.8 another interesting model issued from symmetric spaces.

Theorem 4.10 *Let U be a $2n \times 2n$ random matrix such that $U \stackrel{(d)}{=} g^D g$ where g is Haar distributed on $\mathbb{SO}(2n)$. The spectral measure $\mu_{\mathbf{w}}^{(n)}$ of U is symmetric and the distribution of $(m_1(\gamma_{\mathbf{w}}^{(n)}), \dots, m_{n-1}(\gamma_{\mathbf{w}}^{(n)}))$ is uniform on $M_{n-1}^{[0,1]}$.*

Proof: Assume first that $n = 2N$. From Theorem 4.8 1. it is enough to check that this model induces the distribution $J(4, 1, 1, N) \otimes \text{Dir}_N(2)$. We follow the notation of [12].

Let Φ be the mapping $g \in \mathbb{SO}(2n) \mapsto g^D g$ and let $\mathbf{S}(n)$ be its image. Since $\Phi^{-1}(1) := K(n) = \mathbb{SO}(2n) \cap \text{USp}(2n)$ we can see Φ as a one to one

mapping from $\mathbb{SO}(2n)/K(n)$ onto $\mathbf{S}(n)$. Actually $K(n)$ is isometric to $\mathbb{U}(n)$ via

$$\mathbb{U}(n) \ni g \mapsto \begin{pmatrix} \Re g & -\Im g \\ \Im g & \Re g \end{pmatrix} \in K(n).$$

Let S be a random element of $\mathbf{S}(n)$ whose distribution is the pushforward of $\lambda_{\mathbb{SO}(2n)}$ under Φ . We have

$$S \stackrel{(d)}{=} i(g)a(i(g))^{-1}$$

where g and a are independent, g is $\lambda_{\mathbb{U}(n)}$ distributed and if $n = 2N$

$$a = \begin{pmatrix} \Re \Lambda_N & -\Im \Lambda_N & & \\ \Im \Lambda_N & \Re \Lambda_N & & \\ & & \Re \Lambda_N & \Im \Lambda_N \\ & & -\Im \Lambda_N & \Re \Lambda_N \end{pmatrix}$$

where $\Lambda_N = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$ and the θ_k 's have a joint density proportional to

$$\Delta(\cos \theta_1, \dots, \cos \theta_N)^4 \prod_{k=1}^N |\sin \theta_k|.$$

(see [12], proof of Theorem 2 (formulae 47 and 48)). In terms of $x_k = (1 + \cos \theta_k)/2$ this yields exactly the $J(4, 1, 1, N)$ distribution. Moreover, a simple computation gives

$$\langle e_1, S^j e_1 \rangle = \sum_{r=1}^N \mathbf{w}_r \cos(j\theta_r)$$

with $\mathbf{w}_r = (|g_{1,r}|^2 + |g_{1,N+r}|^2)$ and $(\mathbf{w}_1, \dots, \mathbf{w}_N)$ is $\text{Dir}_N(2)$ distributed.

If $n = 2N + 1$, similar considerations may be made to interpret formula (35) as the random spectral measure of $\mathbb{SO}(2n)/K(n)$ when $n = 2N + 1$ (in that case 1 is double eigenvalue of S). \blacksquare

5 Large deviations for random spectral measures

We present here the LDP for sequences of random measures defined in the previous section. The main result is Theorem 5.3 which is obtained as

a consequence of our generic Proposition 5.2 concerning weighted random measures. This proposition states that the LDP for empirical measures with speed N^2 can be used to derive the LDP with speed N for measures with Dirichlet distributed weights.

For the sake of completeness we briefly recall the LDP definition. Let (u_n) be a decreasing positive sequence of real numbers with $\lim_{n \rightarrow \infty} u_n = 0$.

Definition 5.1 *We say that a sequence (R_N) of probability measures on a measurable Hausdorff space $(G, \mathcal{B}(G))$ satisfies the LDP with rate function I and speed (u_N^{-1}) if:*

- i) *I is lower semicontinuous (lsc), with values in $\mathbb{R}^+ \cup \{+\infty\}$.*
- ii) *For any measurable set A of G :*

$$-I(\text{int } A) \leq \liminf_{N \rightarrow \infty} u_N \log R_N(A) \leq \limsup_{N \rightarrow \infty} u_N \log R_N(A) \leq -I(\text{clo } A),$$

where $I(A) = \inf_{\xi \in A} I(\xi)$ and $\text{int } A$ (resp. $\text{clo } A$) is the interior (resp. the closure) of A .

We say that the rate function I is good if its level set $\{x \in G : I(x) \leq a\}$ is compact for any $a \geq 0$. More generally, a sequence of G -valued random variables is said to satisfy the LDP if the sequence of their distributions satisfies the LDP.

Hereafter, the space G is the set of all non negative measures (or probability measures) supported by a given compact set. We endow this set with the topology of the weak convergence. The rate function obtained in this paper is the so-called reversed Kullback information. Let us recall that if P and Q be probability measures on G , the Kullback information between P and Q is defined by

$$\mathcal{K}(P|Q) = \begin{cases} \int_G \log \frac{dP}{dQ} dP & \text{if } P \ll Q \text{ and } \log \frac{dP}{dQ} \in L^1(P), \\ +\infty & \text{otherwise.} \end{cases} \quad (36)$$

For fixed Q , the nonnegative convex function $\mathcal{K}(\cdot|Q)$ is the rate function of the LDP for the empirical distribution of a sample according to Q (Sanov's theorem [9] p. 263). For fixed P , the function $\mathcal{K}(P|\cdot)$ is nonnegative convex, it vanishes only for $Q = P$, and we call it reversed Kullback information with respect to P .

5.1 LDP for random measures with Dirichlet weights

Proposition 5.2 *Let $\rho > 1$ and $\{\xi_{k,N}, 1 \leq k \leq N\}_{N \geq 1}$ be a triangular array of random variables belonging to a compact interval $[\chi_1, \chi_2]$ and let*

$$\mathcal{L}_N := \frac{1}{N} \sum_{k=1}^N \delta_{\xi_{k,N}}.$$

Assume that the sequence (\mathcal{L}_N) satisfies the LDP in $\mathcal{M}_1([\chi_1, \chi_2])$ with speed (N^ρ) with a good rate function I_ξ . Assume further that I_ξ has a unique minimum at ν whose support is $[\chi_1, \chi_2]$. Let $\{\mathbf{w}_{k,N}, 1 \leq k \leq N\}_{N \geq 1}$ another triangular array, independent of the first one, such that for $N \geq 1$, the vector $(\mathbf{w}_{k,N}, 1 \leq k \leq N)$ follows the distribution $\text{Dir}_N(a)$ ($a > 0$). Then, the family

$$\mu_N := \sum_{k=1}^N \mathbf{w}_{k,N} \delta_{\xi_{k,N}} \tag{37}$$

satisfies the LDP in $\mathcal{M}_1([\chi_1, \chi_2])$ with speed (N) and good rate function is $a\mathcal{K}(\nu|\cdot)$.

Proof:

The proof uses heavily the classical representation (7) :

$$\{\mathbf{w}_{k,N}, 1 \leq k \leq N\} \stackrel{(d)}{=} \left(\frac{Y_1}{Y_1 + \dots + Y_N}, \dots, \frac{Y_N}{Y_1 + \dots + Y_N} \right) \tag{38}$$

where the Y 's are independent and γ_a distributed. The modified measure

$$\tilde{\mu}_N := \frac{1}{N} \sum_{k=1}^N Y_k \delta_{\xi_{k,N}},$$

has then independent weights and is easier to handle. We come back to the original measure with

$$\mu_{\mathbf{w}}^{(N)} = \frac{\tilde{\mu}_N}{\tilde{\mu}_N(1)}.$$

Now, recall that the cumulant generating function of the γ_a distribution is

$$L_Y(\tau) := \begin{cases} a \log(1 - \tau) & \text{if } \tau < 1, \\ +\infty & \text{otherwise,} \end{cases}$$

and that its Cramér transform is (see for example [9])

$$j_a(x) = \begin{cases} x - a - a \log \frac{x}{a} & \text{if } x > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

To understand the flavor of our proof, let us first assume that the sequence $(\xi_{k,N})$ is not random, and that the sequence (\mathcal{L}_N) converges weakly to ν . In that case, using the main Theorem of [30] we see that the sequence $\tilde{\mu}_N$ satisfies the LDP with speed (N) and good rate function :

$$\tilde{I}(\mu) := \int_{[\chi_1, \chi_2]} (j_a(g) - g) d\nu + \int_{[\chi_1, \chi_2]} d\mu \quad (\mu \in \mathcal{M}([\chi_1, \chi_2])), \quad (39)$$

where $d\mu = g d\nu + (\mu - g d\nu)$ is the Lebesgue decomposition of μ with respect to ν . Further, as almost surely $\tilde{\mu}_N([\chi_1, \chi_2]) > 0$ and using the contraction principle (see [9] p.126), we may conclude that (μ_N) satisfies the LDP with speed (N) and good rate function

$$I(\xi) = \inf \{ \tilde{I}(\mu) : \mu \in \mathcal{M}([\chi_1, \chi_2]), \mu = \mu([\chi_1, \chi_2])\xi \} = \inf_{b>0} \tilde{I}(b\xi).$$

Therefore, a direct computation yields

$$I(\xi) = a\mathcal{K}(\nu|\xi).$$

Let us now turn to the general case. We have to get rid of the randomness of the $\xi_{k,N}$. But, we show that, since the LDP with larger speed holds for its empirical measure, this randomness has no effect on the rate function. To get the LDP, we use a method inspired by the proof of the so-called Gärtner-Ellis-Baldi theorem ([9] p.157). Roughly speaking, it consists of two steps. First we compute the limiting normalized generating function and its convex conjugate function. Then we conclude by density of exposed points of this last function.

To begin, let f be a continuous function on \mathbb{R} such that $L_Y \circ f$ is bounded. Then, integrating first on the random variables Y we may write

$$I_N := \mathbb{E} \exp \tilde{\mu}_N(f) = \mathbb{E} \exp \left[\sum_k L_Y(f(\xi_{k,N})) \right] = \mathbb{E} \exp (N \mathcal{L}_N(L_Y \circ f)).$$

Now, from the assumption, the sequence of random variables $(\mathcal{L}_N(L_Y \circ f))$ converges to $\int (L_Y \circ f) d\nu$ and satisfies the LDP with speed (N^ρ) . Now fix $\varepsilon > 0$ and set

$$A := \{ \eta \in \mathcal{M}_+([\chi_1, \chi_2]) : |\eta(L_Y \circ f) - \nu(L_Y \circ f)| \leq \varepsilon \},$$

and $I_N = I_{N,A} + I_{N,A^c}$ with

$$I_{N,A} := \mathbb{E} \left[\exp \left(N \mathcal{L}_N(L_Y \circ f) \right) \mathbf{1}_{\{\mathcal{L}_N \in A\}} \right]. \quad (40)$$

Obviously, we have

$$\exp[N(\nu(L_Y \circ f) - \varepsilon)] \mathbb{P}(\mathcal{L}_N \in A) \leq I_{N,A} \leq \exp[N(\nu(L_Y \circ f) + \varepsilon)] \quad (41)$$

and

$$I_{N,A^c} \leq \exp[N\|L_Y \circ f\|_\infty] \mathbb{P}(\mathcal{L}_N \in A^c) \quad (42)$$

The LDP upper bound gives

$$\limsup \frac{1}{N^\rho} \log \mathbb{P}(\mathcal{L}_N \in A^c) \leq -\inf\{I_\xi(\eta); \eta \in \overline{A^c}\}. \quad (43)$$

Since I_ξ is good, its infimum on the closed set $\overline{A^c}$ is reached and since it has a unique global minimum at ν which is not in $\overline{A^c}$, the right side of (43) is negative. Since $\rho > 1$ we get $\limsup_N \frac{1}{N} \log I_{N,A^c} = -\infty$, and then

$$\limsup_N \frac{1}{N} \log I_N \leq \nu(L_Y \circ f) + \varepsilon. \quad (44)$$

To get the lower bound, we have

$$\liminf_N \frac{1}{N} \log I_N \geq \liminf_N \frac{1}{N} \log I_{N,A} \geq \nu(L_Y \circ f) - \varepsilon + \liminf_N \frac{1}{N} \log \mathbb{P}(\mathcal{L}_N \in A)$$

From (43) it is clear that $\lim \mathbb{P}(\mathcal{L}_N \in A) = 1$ so that

$$\liminf_N \frac{1}{N} \log I_N \geq \nu(L_Y \circ f) - \varepsilon \quad (45)$$

As (44) and (45) hold for every $\varepsilon > 0$, we may conclude

$$\lim_N \frac{1}{N} \log I_N = \nu(L_Y \circ f).$$

The remaining of the proof is the same as the LDP proofs developed in [17] (see also [10]). ■

5.2 Large deviations for spectral measures in the circular and Jacobi ensembles

We now state the LDP for the sequence of random probability measures $(\mu_{\mathbf{w}}^{(N)})$ in the above frameworks. Notice that in the case $\beta = 2$, with the help of Theorem 4.1, the first result has yet been shown in [27] and the second one in [15] Theorem 2.3. In this last paper, the proofs use the so-called projective limit approach. We take here a completely different way. Indeed, we give a general proof by using a variation on the results developed in [30] for random measures. As a matter of fact, we have shown in Subsection 5.1 the LDP for random measures with weights having a Dirichlet distribution.

Theorem 5.3

1. Assume that for any N , $\mathbb{T}^N \times \mathcal{S}_N$ is endowed with the $C\beta E_N$ distribution. Recall that $\mu_{\mathbf{w}}^{(N)}$ is built as in (23). Then, the sequence of random probability distributions $(\mu_{\mathbf{w}}^{(N)})$ satisfies the LDP with speed N and good rate function

$$I(\xi) = \frac{\beta}{2} \mathcal{K}(\lambda_{\mathbb{T}}|\xi), \quad (\xi \in \mathcal{M}_1(\mathbb{T})) \quad (46)$$

where $\lambda_{\mathbb{T}}$ is the uniform distribution on \mathbb{T} .

2. Let $a, b > 0$ be fixed. The sequence $(\xi^{(N)})$ under the $J(\beta, a, b, N) \otimes \text{Dir}_N(\beta/2)$ distribution satisfies the LDP with speed N and good rate function

$$I(\xi) = \frac{\beta}{2} \mathcal{K}(\arcsine|\xi) \quad (47)$$

Proof: The proof of this theorem is a direct consequence of Proposition 5.2. Indeed, from [20] it is known that the empirical distribution built on (θ_k) satisfies the LDP with speed N^2 whose rate function has a unique minimum in $\lambda_{\mathbb{T}}$. For the Jacobi ensemble, Hiai and Petz ([22] Section 2) proved the LDP with speed N^2 for the empirical measure built on (x_k) . Moreover, they have shown that - with our assumptions - the unique minimizer of the rate function for this LDP is the arcsine distribution.

Remark 5.4 *The same conclusion holds true under the probability measures of Theorem 4.8.*

Remark 5.5 *The assumptions about the supports are crucial. In a companion paper [16] we study important models in which they are not satisfied. For the GUE_N , the support of \mathcal{L}_N is not included in a fixed compact set and the limiting distribution is the semicircle. In this example, we find an additional term in the rate function. For the $J(\beta, \tau_1 N, \tau_2 N; N)$, the support of \mathcal{L}_N is included in $[0, 1]$ but the limiting distribution is supported by a compact subinterval of $(0, 1)$. It is called the Kesten-MacKay distribution and generalizes the arcsine distribution. In [16], we prove the LDP but the explicit form of the rate function is only conjectured. The same is true for the Laguerre ensemble.*

Remark 5.6 *Other discrete random measures with Dirichlet weights appear in the literature. Let α be a positive measure on a compact space K . A Dirichlet process μ of parameter α is a probability distribution denoted by $\mathcal{D}(\alpha)$ on $\mathcal{M}_1(K)$. It is such that for any measurable finite partition of K , (A_1, \dots, A_N) , we have $(\mu(A_1), \dots, \mu(A_N)) \stackrel{(d)}{=} \text{Dir}_N(\alpha(A_1), \dots, \alpha(A_N))$. If α is the finite discrete measure $\theta \sum_{k=1}^N \delta_{y_k}$ where $y_1, \dots, y_N \in K$, we have*

$$\mathcal{D}\left(\theta \sum_{j=1}^N \delta_{y_j}\right) \stackrel{(d)}{=} \sum_{k=1}^N Y_k \delta_{y_k} \quad (48)$$

where $(Y_1, \dots, Y_N) \stackrel{(d)}{=} \text{Dir}_N(\theta)$. This means that the random measures studied in the previous sections are mixed Dirichlet processes. The mixing distribution is the law of the random eigenvalues. Ganesh and O'Connell ([18]) study $\mathcal{D}(\alpha + \sum_{i=1}^N \delta_{y_i})$ when $N^{-1} \sum_{i=1}^N \delta_{y_i}$ converges to some measure ν and when α is a measure whose support is K . They show that μ satisfies the LDP with speed N , and good rate function $I(\cdot) = \mathcal{K}(\nu|\cdot)$. In [7] there is an extension with an infinite number of random locations, (see also [8] for the connection with the Poisson-Dirichlet distribution).

5.3 Relation with spherical integrals

In the unitary model, the direct computation of the limiting cumulant generating functional of $\mu_{\mathbf{w}}^{(N)}$ leads to a spherical integral. Indeed, if $\varphi \in C(\mathbb{T})$ (the set of all continuous function on \mathbb{T}), we have

$$\mu_{\mathbf{w}}^{(N)}(\varphi) = \langle e_1, \varphi(U) e_1 \rangle = \text{tr}(G_N V D_{\varphi}^{(N)} V^*)$$

where $D_\varphi^{(N)} := \text{diag}(\varphi(e^{i\theta_1}), \dots, \varphi(e^{i\theta_N}))$, (the $e^{i\theta_k}$ are the eigenvalues of U), $V \in \mathbb{U}(N)$ and $G_N := \text{diag}(1, 0, \dots, 0)$. Consequently, the Laplace transform of $\mu_{\mathbf{w}}$ is, for $\varphi \in C(\mathbb{T})$,

$$\mathbb{E} \exp(N\mu_{\mathbf{w}}^{(N)}(\varphi)) = \mathbb{E}_\theta I_N^{(2)}(G_N, D_\varphi^{(N)}) \quad (49)$$

where

$$I_N^{(2)}(G_N, D_\varphi^{(N)}) = \left(\int_{\mathbb{U}(N)} \exp[N \text{tr}(G_N V D_\varphi^{(N)} V^*)] \, dV \right),$$

(\mathbb{E}_θ denotes here expectation with respect to the variables $\theta_1, \dots, \theta_N$). It is interesting to notice that this last spherical integral over the unitary group can be expressed as a hypergeometric function with two matrix arguments (see for example [31] p. 97).

In [19] and [6] it is proved that $N^{-1} \log I_N^{(2)}(G_N, D_\varphi^{(N)})$ has a limit as $N \rightarrow \infty$, as soon as the empirical spectral distribution of $D_\varphi^{(N)}$ converges weakly. More precisely in [19] Th.6 (see also [6] Th. 4) it is proved that

$$\lim_N \frac{1}{N} \log I_N^{(2)}(G_N, D_\varphi^{(N)}) = F_\varphi(1), \quad (50)$$

where $F_\varphi(1)$ is defined in the following way. Let ν_φ be the limit of the empirical spectral distribution of $D_\varphi^{(N)}$, (in our case it is the image by φ of $\Lambda_{\mathbb{T}}$). Further let $\varphi_{\min} := \min_{z \in \mathbb{T}} \varphi(z)$ and $\varphi_{\max} := \max_{z \in \mathbb{T}} \varphi(z)$. The Stieltjes transform of ν_φ is defined for $x \in (-\infty, \varphi_{\min}) \cup (\varphi_{\max}, +\infty)$ by:

$$H_\varphi(x) = \int_{\mathbb{T}} \frac{1}{x - \varphi(z)} \lambda_{\mathbb{T}}(dz). \quad (51)$$

We have that $H_\varphi^\downarrow := \lim_{x \downarrow \varphi_{\max}} H_\varphi(x) < 0 < H_\varphi^\uparrow := \lim_{x \uparrow \varphi_{\min}} H_\varphi(x)$ and the range of H_φ is $(H_\varphi^\downarrow, 0) \cup (0, H_\varphi^\uparrow)$. ([19] Property 9). The Voiculescu R -transform is the function R_φ satisfying, for all y in the range of H_φ ,

$$H_\varphi \left(R_\varphi(y) + \frac{1}{y} \right) = y. \quad (52)$$

As a result ([19] Theorem 6),

$$\begin{aligned} F_\varphi(1) &= v(1) - \int_{\mathbb{T}} \log(1 + v(1) - \varphi(z)) \, \lambda_{\mathbb{T}}(dz) \\ v(1) &= \begin{cases} R_\varphi(1) & \text{if } H_\uparrow \leq 1 \leq H_\downarrow \\ \varphi_{\max} - 1 & \text{if } 1 > H_\downarrow \\ \varphi_{\min} - 1 & \text{if } 1 < H_\uparrow \end{cases} \end{aligned} \quad (53)$$

Now, the LDP for the empirical distribution built on $\theta_1, \dots, \theta_N$ holds with speed N^2 . So that, $N^{-1} \ln \mathbb{E} \exp (N \mu_w^{(N)}(\varphi))$ and $N^{-1} \log I_N^{(2)}(G_N, D_\varphi^{(N)})$ have the same limit (to show this claim just use the continuity of spherical integrals proved in [28] Prop. 2.1). The rate function of the LDP for $\mu_w^{(N)}$ can be recovered by taking the supremum in $\varphi \in C(\mathbb{T})$ of $\mu(\varphi) - F_\varphi(1)$, where $\mu \in \mathcal{M}_1(\mathbb{T})$. Setting $g(e^{i\theta}) = \varphi(e^{i\theta}) - v(1)$, we have

$$\mu(\varphi) - F_\varphi(1) = \int_{\mathbb{T}} g(z) \mu(dz) + \int_{\mathbb{T}} \log(1 - g(z)) \lambda_{\mathbb{T}}(dz). \quad (54)$$

Taking the supremum in $g \in C(\mathbb{T})$, we recover the well-known duality formula

$$\sup_{g \in C(\mathbb{T})} \left[\int_{\mathbb{T}} g(z) d\mu(z) + \int_{\mathbb{T}} \log(1 - g(z)) \lambda_{\mathbb{T}}(dz) \right] = \mathcal{K}(\lambda_{\mathbb{T}} | \mu).$$

References and several consequences of this formula for the associated moment problem may be found in [15].

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